

$F(T)$ gravity and k-essence

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Abstract

Modified teleparallel gravity theory with the torsion scalar have recently gained a lot of attention as a possible explanation of dark energy. We perform a thorough reconstruction analysis on the so-called $F(T)$ models, where $F(T)$ is some general function of the torsion term, and derive conditions for the equivalence between of $F(T)$ models with purely kinetic k-essence. We present a new class models of $F(T)$ -gravity and k-essence. We also proposed some new models of generalized gases and knot universes as well as some generalizations of $F(T)$ gravity.

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1 Introduction

The discovery of the accelerated expansion of the universe [1] has forced a profound shift in our cosmological paradigm. This discovery indicates that the universe is very nearly spatially flat and consists of about 70% of dark energy (DE) which drives the cosmic acceleration. The equation of state (EoS) parameter w for DE should be $w < -1/3$ to maintain this acceleration. The modern constraints on the EoS parameter are around $w = -1$. Here we can note that from the theoretical point of view there are three essentially different cases: $w > -1$ (quintessence), $w = -1$ (cosmological constant), and $w < -1$ (phantom). The cosmological constant can explain the present accelerated expansion of the universe, for which $w = -1$. Although, the cosmological constant is the simplest candidate for DE, but there are serious theoretical problems associated with it such as the fine-tuning problem, the coincidence problem and so on. To solve the cosmological constant problems, some scalar-field models (phantom fields, k-essence and so on) are proposed. These scalar-field models of inflation and dark energy correspond to a modification of the energy momentum tensor in Einstein equations.

The other alternative approach dealing with the acceleration problem of the universe is changing the gravity law through the modification of action of gravity by means of using $F(R)$, $F(G)$ and $F(R, G)$ instead of the Einstein-Hilbert action (see, e.g. recent reviews [2]-[13]). Here the Lagrangian density of modified gravity theories F is an arbitrary function of R , G or both R and G . The field equations of these modified gravity theories are 4th order that making it difficult to obtain both exact and numerical solutions.

Recently, however, some models with the field equations of 2nd order [so-called $F(T)$ -gravity] are proposed [14]-[15]. These models based on the "teleparallel" equivalent of General Relativity (TEGR) [16]-[22], which, instead of using the curvature defined via the Levi-Civita connection, uses the Weitzenböck connection that has no curvature but only torsion. In [21]-[22], some models based on modified teleparallel gravity were presented as an alternative to inflationary models. The fact that the field equations of $F(T)$ gravity are always 2nd order makes these theories simpler than the other modified gravity theories like $F(R)$ or $F(G)$. More recently, some properties of $F(T)$ gravity were studied in [23]-[33]. It is clear that $F(T)$ gravity presents a very rich behavior and deserves further investigation.

The purpose of the present paper is to investigate some models of $F(T)$ gravity as well as k-essence. Also we will study the equivalence of modified gravity theories with k-essence.

This paper is organized as follows. In the following section we review $F(T)$ gravity and present some its models. In Sec. III we investigate some models of k-essence. The relation between $F(T)$ gravity and k-essence is studied in Sec.IV. In the last section we will give some conclusions.

2 $F(T)$ gravity

2.1 Elements of $F(T)$ gravity

The action of $F(T)$ - gravity reads as (see, e.g. [14], [15], [23])

$$S = \int d^4x e \left[\frac{1}{2\kappa^2} F(T) + L_m \right], \quad (2.1)$$

where T is the torsion scalar, $e = \det(e_\mu^i) = \sqrt{-g}$ and L_m stands for the matter Lagrangian. Here e_μ^i are the components of the vierbein vector field \mathbf{e}_A in a coordinate basis, that is $\mathbf{e}_A \equiv e_A^\mu \partial_\mu$. Note that in the teleparallel gravity, the dynamical variable is the vierbein field $\mathbf{e}_A(x^\mu)$. The variation of the action with respect to this vierbein field leads to the following gravitational equations of motion

$$[e^{-1} \partial_\mu (e S_i^{\mu\nu}) - e_i^\lambda T_{\mu\lambda}^\rho S_\rho^{\nu\mu}] F_T + S_i^{\mu\nu} (\partial_\mu T) F_{TT} + \frac{1}{4} e_i^\nu F = \frac{1}{2} k^2 e_i^\rho T_\rho^\nu. \quad (2.2)$$

Here the torsion scalar T is given by

$$T = S_\rho^{\mu\nu} T_{\mu\nu}^\rho \quad (2.3)$$

with

$$S_\rho^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\theta\nu}{}_\theta - \delta_\rho^\nu T^{\theta\mu}{}_\theta). \quad (2.4)$$

Here the contorsion tensor is defined as

$$K^{\mu\nu}{}_\rho = -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}) \quad (2.5)$$

and the torsion tensor looks like

$$T_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda = e_i^\lambda (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i). \quad (2.6)$$

The vierbein vector fields relate with the metric through

$$g_{\mu\nu}(x) = \eta_{ij} e_\mu^i(x) e_\nu^j(x), \quad (2.7)$$

where $\mathbf{e}_i \cdot \mathbf{e}_j = \eta_{ij}$ and $\eta_{ij} = \text{diag}(1, -1, -1, -1)$. We now will assume a flat homogeneous and isotropic FRW universe with the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2, \quad (2.8)$$

where t is cosmic time. Then the modified Friedmann equations and the continuity equation read as (see, e.g. [14], [15], [23])

$$-2TF_T + F = 2k^2 \rho_m, \quad (2.9)$$

$$-8\dot{H}TF_{TT} + (2T - 4\dot{H})F_T - F = 2k^2 p_m, \quad (2.10)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (2.11)$$

This set can be rewritten as

$$-T - 2Tf_T + f = 2k^2 \rho_m, \quad (2.12)$$

$$-8\dot{H}Tf_{TT} + (2T - 4\dot{H})(1 + f_T) - T - f = 2k^2 p_m, \quad (2.13)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (2.14)$$

with the action

$$S = \int d^4x e \left[\frac{1}{2\kappa^2} (T + f(T)) + L_m \right], \quad (2.15)$$

where $f = F - T$. Some properties of $F(T)$ - gravity were studied in [15]-[33]. Note that we can rewrite the gravitational equations (2.9)-(2.10) as

$$\hat{M}_1 F = 2k^2 \rho_m, \quad (2.16)$$

$$\hat{M}_2 F = -\hat{M}_3 \hat{M}_1 F = 2k^2 p_m, \quad (2.17)$$

$$\hat{M}_3 \rho_m = -p_m, \quad (2.18)$$

where

$$\hat{M}_1 = -2T\partial_T + 1, \quad (2.19)$$

$$\hat{M}_2 = -8\dot{H}T\partial_{TT}^2 + (2T - 4\dot{H})\partial_T - 1 = (4\dot{H}\partial_T - 1)\hat{M}_1 = -(\frac{1}{3H}\partial_t + 1)\hat{M}_1 = -\hat{M}_3\hat{M}_1, \quad (2.20)$$

$$\hat{M}_3 = \frac{1}{3H}\partial_t + 1. \quad (2.21)$$

Using these basic equations we can construct high hierarchy of $F(T)$ gravity. For the case $\rho_m = p_m = 0$ such hierarchy can be written as

$$\hat{M}_1^n F_n = 0, \quad (2.22)$$

where $F_1 = F$. Some equations from this hierarchy for $n = 1, 2, 3, \dots$ are

$$-2TF_{1T} + F_1 = 0, \quad (2.23)$$

$$4T^2 F_{2TT} + F_2 = 0, \quad (2.24)$$

$$-8T^3 F_{3TTT} - 12T^2 F_{3TT} - 2TF_{3T} + F_3 = 0, \quad (2.25)$$

and so on. From the system (2.16)-(2.18) follows that any solution of the equation (2.16) automatically solves the equations (2.17)-(2.18). It means that we need just to solve the equation (2.16), as that guarantees a solution to the equations (2.17) and (2.18). Finally we present the effective EoS parameter

$$w_{eff} = -1 - 3^{-1}H^{-1}[\ln(\hat{M}_1 F)]_t = -1 - 3^{-1}[\ln(\hat{M}_1 F)]_N. \quad (2.26)$$

2.2 Particular models of $F(T)$ gravity

We note that some explicit models of $F(T)$ gravity appeared in the literature (see, e.g. [14],[15], [23], [24], [27], [28], [31]). Here we would like to present some new models of modified teleparallel gravity.

2.2.1 Example 1: The M_{13} - model

Let us consider the M_{13} - model. Its Lagrangian is

$$F(T) = \sum_{j=-m}^n \nu_j(t)T^j = \nu_{-m}(t)T^{-m} + \dots + \nu_{-1}(t)T^{-1} + \nu_0(t) + \nu_1(t)T + \dots + \nu_n(t)T^n. \quad (2.27)$$

Consider the particular example when $m = n = 1$ and $\nu_j = \text{consts.}$ Then

$$F = \nu_{-1}T^{-1} + \nu_0 + \nu_1T, \quad F_T = -\nu_{-1}T^{-2} + \nu_1, \quad F_{TT} = 2\nu_{-1}T^{-3}. \quad (2.28)$$

Substituting these expressions into (2.9)-(2.10) we obtain

$$3k^{-2}H^2 = \rho_{eff} + \rho_m, \quad (2.29)$$

$$-k^{-2}(2\dot{H} + 3H^2) = p_{eff} + p_m, \quad (2.30)$$

where

$$\rho_{eff} = k^{-2}[3H^2 - 1.5\nu_{-1}T^{-1} + 0.5\nu_1T - 0.5\nu_0], \quad (2.31)$$

$$p_{eff} = k^{-2}[6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} - 0.5\nu_1T + 0.5\nu_0 + 2(\nu_1 - 1)\dot{H} - 3H^2]. \quad (2.32)$$

The effective EoS parameter is given by

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}} = \frac{6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} - 0.5\nu_1T + 0.5\nu_0 + 2(\nu_1 - 1)\dot{H} - 3H^2}{3H^2 - 1.5\nu_{-1}T^{-1} + 0.5\nu_1T - 0.5\nu_0}. \quad (2.33)$$

Let us set $\nu_1 = 1$. Then

$$\rho_{eff} = k^{-2}[-1.5\nu_{-1}T^{-1} - 0.5\nu_0], \quad p_{eff} = k^{-2}[6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} + 0.5\nu_0] \quad (2.34)$$

and

$$w_{eff} = \frac{p_{eff}}{\rho_{eff}} = \frac{6\nu_{-1}\dot{H}T^{-2} + 1.5\nu_{-1}T^{-1} + 0.5\nu_0}{-1.5\nu_{-1}T^{-1} - 0.5\nu_0} = -1 - \frac{6\nu_{-1}\dot{H}T^{-2}}{1.5\nu_{-1}T^{-1} + 0.5\nu_0}, \quad (2.35)$$

respectively.

2.2.2 Example 2: The M_{21} - model

Our next example is the M_{21} - model

$$F = T + \alpha T^\delta \ln T. \quad (2.36)$$

Then

$$F_T = 1 + \alpha\delta T^{\delta-1} \ln T + \alpha T^{\delta-1}, \quad F_{TT} = \alpha\delta(\delta-1)T^{\delta-2} \ln T + \alpha(2\delta-1)T^{\delta-2}. \quad (2.37)$$

In this case, Eqs.(2.9)-(2.10) take the form

$$-T - 2\alpha T^\delta - \alpha(2\delta-1)T^\delta \ln T = 2k^2\rho_m, \quad (2.38)$$

$$\alpha(2\delta-1)(T - 4\delta\dot{H})T^{\delta-1} \ln T + T - 4\dot{H} + 2\alpha T^\delta - 4\alpha\dot{H}(4\delta-1)T^{\delta-1} = 2k^2p_m. \quad (2.39)$$

So we have

$$\rho_{eff} = 0.5k^{-2}[2\alpha T^\delta + \alpha(2\delta-1)T^\delta \ln T], \quad (2.40)$$

$$p_{eff} = -0.5k^{-2}\alpha T^{\delta-1}[(2\delta-1)(T - 4\delta\dot{H}) \ln T + 2T - 4(4\delta-1)\dot{H}]. \quad (2.41)$$

The special case $\delta = 0.5$ deserves separate consideration. In this case the above equations take the more simple form

$$-T - 2\alpha T^{0.5} = 2k^2\rho_m, \quad T - 4\dot{H} + 2\alpha T^{0.5} - 4\alpha\dot{H}T^{-0.5} = 2k^2p_m. \quad (2.42)$$

For the density of energy and pressure we get the following expressions

$$\rho_{eff} = k^{-2}\alpha T^{0.5}, \quad p_{eff} = -k^{-2}\alpha T^{-0.5}(T - 2\dot{H}). \quad (2.43)$$

2.2.3 Example 3: The M_{22} - model

Now we consider the M_{22} - model

$$F = T + f(y), \quad y = \tanh[T]. \quad (2.44)$$

Then

$$F_T = 1 + f_y(1 - y^2), \quad F_{TT} = f_{yy}(1 - y^2)^2 - 2y(1 - y^2)f_y \quad (2.45)$$

so that Eqs.(2.9)-(2.10) take the form

$$-T - 2(1 - y^2)Tf_y + f = 2k^2\rho_m, \quad (2.46)$$

$$T - 4\dot{H} - 8(1 - y^2)^2 T \dot{H} f_{yy} + (16y\dot{H}T + 2T - 4\dot{H})(1 - y^2)f_y - f = 2k^2 p_m. \quad (2.47)$$

So we have

$$\rho_{eff} = 0.5k^{-2}[2(1 - y^2)Tf_y - f], \quad (2.48)$$

$$p_{eff} = 0.5k^{-2}[8(1 - y^2)^2 T \dot{H} f_{yy} - (16y\dot{H}T + 2T - 4\dot{H})(1 - y^2)f_y + f]. \quad (2.49)$$

The EoS parameter reads as

$$\begin{aligned} w_{eff} &= \frac{8(1 - y^2)^2 T \dot{H} f_{yy} - (16y\dot{H}T + 2T - 4\dot{H})(1 - y^2)f_y + f}{2(1 - y^2)Tf_y - f} = \\ &= -1 + \frac{8(1 - y^2)^2 T \dot{H} f_{yy} - (16y\dot{H}T - 4\dot{H})(1 - y^2)f_y + f}{2(1 - y^2)Tf_y - f}. \end{aligned} \quad (2.50)$$

2.2.4 Example 4: The M_{25} - model

In this subsection we consider the M_{25} - model

$$F = \sum_{-m}^n \nu_j(t) \xi^j, \quad (2.51)$$

where $\xi = \ln T$. As an example we consider the case $m = n = 1$, $\nu_j = \text{consts}$ that is

$$F = \nu_{-1} \xi^{-1} + \nu_0 + \nu_1 \xi. \quad (2.52)$$

Then

$$F_\xi = -\nu_{-1} \xi^{-2} + \nu_1, \quad F_{\xi\xi} = 2\nu_{-1} \xi^{-3} \quad (2.53)$$

and

$$F_T = (-\nu_{-1} \xi^{-2} + \nu_1) e^{-\xi}, \quad F_{TT} = (2\nu_{-1} \xi^{-3} + \nu_{-1} \xi^{-2} - \nu_1) e^{-2\xi}. \quad (2.54)$$

For this case, Eqs.(2.9)-(2.10) read as

$$2\nu_{-1} \xi^{-2} + \nu_{-1} \xi^{-1} + \nu_0 - 2\nu_1 + \nu_1 \xi = 2k^2 \rho_m, \quad (2.55)$$

$$-4\dot{H}(4\nu_{-1} \xi^{-3} + \nu_{-1} \xi^{-2} - \nu_1) e^{-\xi} - 2\nu_{-1} \xi^{-2} - \nu_{-1} \xi^{-1} + 2\nu_1 - \nu_0 - \nu_1 \xi = 2k^2 p_m. \quad (2.56)$$

3 K-essence

3.1 Elements of k-essence

The action of k-essence has the form [34]-[36]

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + K(X, \phi) + L_m \right]. \quad (3.1)$$

The corresponding closed set of equations for the FRW metric (2.8) reads as

$$3k^{-2}H^2 = 2XK_X - K + \rho_m, \quad (3.2)$$

$$-k^{-2}(2\dot{H} + 3H^2) = K + p_m, \quad (3.3)$$

$$(K_X + 2XK_{XX})\dot{X} + 6HXK_X - K_\phi = 0, \quad (3.4)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (3.5)$$

where $X = -0.5\dot{\phi}^2$. The equation of motion of the scalar field ϕ is given as

$$-(a^3\dot{\phi}K_X)_t = a^3K_\phi, \quad (3.6)$$

which is just the other form of the equation (3.4). In the purely kinetic k-essence case we have $K_\phi = 0$ and from the last equation we get (see, e.g. [37])

$$a^3\dot{\phi}K_X = a^3\sqrt{-2X}K_X = \sqrt{\kappa} = \text{const}. \quad (3.7)$$

3.2 Particular models of k-essence

As examples in this subsection we would like to present some new types of k-essence. We believe that all of them can give rise to cosmic acceleration.

3.2.1 Example 1: The M_{12} - model

Let us consider the M_{12} - model with the following Lagrangian

$$K = \nu_{-m}(N)N^{-m} + \dots + \nu_{-1}(N)N^{-1} + \nu_0(N) + \nu_1(N)N + \dots + \nu_n(N)N^n, \quad (3.8)$$

where in general $\nu_j = \nu_j(\phi) = \nu_j(N)$ and $N = \ln(aa_0^{-1})$. As an example, we study the case $m = 0, n = 2, \nu_j = \text{const}$. In this case, the M_{12} - model becomes

$$K = \nu_0 + \nu_1 N + \nu_2 N^2. \quad (3.9)$$

To find ν_j and X we look for H for example as

$$H = \mu_0 + \mu_1 N, \quad (3.10)$$

where $\mu_j = \text{const}$ [in general $\mu_j = \mu_j(t)$]. Of course

$$a = a_0 e^N. \quad (3.11)$$

Finally, we obtain the following parametric form of the M_{12} - model (parametric purely kinetic k-essence)

$$K = -(2\mu_0\mu_1 + 3\mu_0^2) - 2\mu_1(\mu_1 + 3\mu_0)N - 3\mu_1^2 N^2, \quad (3.12)$$

$$X = k^{-1} a_0^6 \mu_1^2 (\mu_0 + \mu_1 N)^2 e^{6N}. \quad (3.13)$$

3.2.2 Example 2: The M_1 - model

Our next example is the M_1 - model. Its Lagrangian looks like

$$K = \nu_{-m}(t)t^{-m} + \dots + \nu_{-1}(t)t^{-1} + \nu_0(t) + \nu_1(t)t + \dots + \nu_n(t)t^n, \quad (3.14)$$

where in general $\nu_j = \nu_j(\phi) = \nu_j(t)$. Let us explore this model for the case: $m = 0, n = 2$ and $\nu_j = \text{const}$. In this case the M_1 - model takes the form

$$K = \nu_0 + \nu_1 t + \nu_2 t^2. \quad (3.15)$$

To find ν_j and X we look for H , e.g. as

$$H = \mu_0 + \mu_1 t \quad (3.16)$$

so that

$$a = a_0 e^{\mu_0 t + 0.5 \mu_1 t^2}, \quad (3.17)$$

where $\mu_j = \text{const}$ [in general $\mu_j = \mu_j(t)$]. After some calculations we obtain the following explicit form of the k-essence Lagrangian

$$K = -(2\mu_1 + 3\mu_0^2) - 6\mu_0\mu_1 t - 3\mu_1^2 t^2. \quad (3.18)$$

At the same time, we have

$$2XK_X = 3H^2 + K = -2\dot{H} = -2\mu_1. \quad (3.19)$$

For X we get the following expression

$$X = \gamma_2^{-1} e^{6\mu_0 t + 3\mu_1 t^2}, \quad \gamma_2^{-1} = \kappa^{-1} a_0^6 \mu_1^2. \quad (3.20)$$

Hence follows that

$$t = \frac{1}{3\mu_1}[-3\mu_0 \pm \sqrt{9\mu_0^2 + 3\mu_1 \ln(\gamma_2 X)}]. \quad (3.21)$$

Finally, we come to the following M_{23} - model

$$K = -2\mu_1 - 3\mu_0^2 - \mu_1 \ln[\gamma_2 X] = \nu_0 + \nu_1 \ln X. \quad (3.22)$$

[We recall that in general the M_{23} -model reads as

$$K = \nu_{-m}(t)\zeta^{-m} + \dots + \nu_{-1}(t)\zeta^{-1} + \nu_0(t) + \nu_1(t)\zeta + \dots + \nu_n(t)\zeta^n, \quad (3.23)$$

where $\zeta = \ln X$.]

3.2.3 Example 3: The M_{24} - model

Here we present the following M_{24} - model

$$K = \frac{2m\lambda\sigma^2(-2\beta v + \lambda v^2 + \lambda)(1 - v^2)}{(\beta - \lambda v)^2} - 3[n - \frac{m\lambda\sigma(1 - v^2)}{\beta - \lambda v}]^2, \quad (3.24)$$

$$X = \gamma_3(2\beta v - \lambda v^2 - \lambda)^2(1 - v^2)^2(\beta - \lambda v)^{6m-4}, \quad (3.25)$$

where $\gamma_3 = \kappa^{-1}\alpha^6 m^2 \lambda^2 \sigma^6$, $v = \tanh[\sigma t]$ and $\lambda, \sigma, \alpha, \beta, n, m$ are some constants. Solving the equation (3.3) we obtain

$$H = n - \frac{m\lambda\sigma(1 - v^2)}{\beta - \lambda v} \quad (3.26)$$

and hence for the scale factor we get the following formula

$$a = \alpha[\beta - \lambda v]^m e^{nt}. \quad (3.27)$$

Note that

$$\dot{H} = \frac{m\lambda\sigma^2(2\beta v - \lambda v^2 - \lambda)(1 - v^2)}{(\beta - \lambda v)^2}. \quad (3.28)$$

4 Equivalence between $F(T)$ -gravity and k-essence

In this section, our goal is to study the relation between modified teleparallel gravity and purely kinetic k-essence. In Appendix C, we will consider this relation in the context with the other modified gravity theories.

4.1 General case

4.1.1 Variant-I

Consider the transformation

$$K = 8\dot{H}T f_{TT} - 2(T - 2\dot{H})f_T + f, \quad (4.1)$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2, \quad (4.2)$$

where $T = -6H^2$. Then Eqs.(2.12)-(2.14) take the form

$$0 = -3k^{-2}H^2 + 2XK_X - K + \rho_m, \quad (4.3)$$

$$0 = k^{-2}(2\dot{H} + 3H^2) + K + p_m, \quad (4.4)$$

$$(K_X + 2XK_{XX})\dot{X} + 6HXK_X = 0, \quad (4.5)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (4.6)$$

These are the equations of motion of purely kinetic k-essence. This result shows that modified teleparallel gravity and purely kinetic k-essence is equivalent to each other, at least in the equation's level. This equivalence allows us to construct a new class of purely kinetic k-essence models starting from some models of modified teleparallel gravity. Let us demonstrate it for the following modified teleparallel gravity model: $f(T) = \alpha T^n$ [14]-[15]. In this case, we have

$$f_T = \alpha n T^{n-1}, \quad f_{TT} = \alpha n(n-1)T^{n-2}. \quad (4.7)$$

Substituting these expressions into the equations (4.1)-(4.2) we get

$$K = 8\alpha n(n-1)\dot{H}T^{n-1} - 2\alpha n(T - 2\dot{H})T^{n-1} + \alpha T^n, \quad (4.8)$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2. \quad (4.9)$$

i) Let $a = a_0 e^{g(t)}$ so that $H = \dot{g}$, $\dot{H} = \ddot{g}$. In this case, K and X take the form

$$K = 8\alpha n(n-1)\ddot{g}(-6)^{n-1}\dot{g}^{2(n-1)} - 2\alpha n(-6\dot{g}^2 - 2\ddot{g})(-6)^{n-1}\dot{g}^{2(n-1)} + \alpha(-6)^n\dot{g}^{2n}, \quad (4.10)$$

$$X = \kappa^{-1}k^{-4}a^6\ddot{g}^2. \quad (4.11)$$

Now if we consider the simplest case $g = t$ (that means $\dot{g} = 1, \ddot{g} = 0$), then we get

$$K = -2\alpha n(-6)^n + \alpha(-6)^n = (1 - 2n)\alpha(-6)^n, \quad (4.12)$$

$$X = 0. \quad (4.13)$$

ii) The more non-trivial model we get, if we consider the example $a = a_0 t^m$. In this case $H = mt^{-1}$, $\dot{H} = -mt^{-2}$, $T = \frac{-6m^2}{t^2}$ so that K and X take the form

$$K = 8\alpha n(n-1)\dot{H}\left(\frac{-6m^2}{t^2}\right)^{n-1} - 2\alpha n\left(\frac{-6m^2}{t^2} - 2\dot{H}\right)\left(\frac{-6m^2}{t^2}\right)^{n-1} + \alpha\left(\frac{-6m^2}{t^2}\right)^n, \quad (4.14)$$

$$X = \kappa^{-1}k^{-4}a_0^6 m^2 t^{6m-4} \quad (4.15)$$

or

$$K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1-3m) + 3m]t^{-2n}, \quad (4.16)$$

$$X = \kappa^{-1}k^{-4}a_0^6 m^2 t^{6m-4} = \gamma_5^{-1}t^{6m-4}. \quad (4.17)$$

Since $t = (\gamma_5 X)^{\frac{1}{6m-4}}$ finally we get the following purely kinetic k-essence model

$$K = 2\alpha m(-6m^2)^{n-1}[-4n(n-1) + 2n(1-3m) + 3m](\gamma_5 X)^{\frac{n}{2-3m}}. \quad (4.18)$$

4.1.2 Variant-II

Let us rewrite Eqs.(2.12)-(2.14) as

$$3k^{-2}H^2 = \rho_{eff} + \rho_m, \quad (4.19)$$

$$-k^{-2}(2\dot{H} + 3H^2) = p_{eff} + p_m, \quad (4.20)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (4.21)$$

where

$$\rho_{eff} = 2Tf_T - f, \quad p_{eff} = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f. \quad (4.22)$$

We now introduce two functions K and X as

$$K = 8\dot{H}Tf_{TT} - 2(T - 2\dot{H})f_T + f, \quad X = \frac{4\dot{H}^2(2Tf_{TT} + f_T)^2}{\kappa a^{-6}}. \quad (4.23)$$

Clearly that these two functions K and X obey the system of the equations (4.3)-(4.6).

4.2 Special case: $\phi = \phi_0 + \ln a^{\pm\sqrt{12}}$

One of the interesting special cases is when:

$$\phi = \phi_0 + \ln a^{\pm\sqrt{12}}. \quad (4.24)$$

It deserves separate investigation. In fact for this case $\dot{\phi} = \pm\sqrt{12}H$ so that $X = -0.5\dot{\phi}^2 = -6H^2 = T$. Then the corresponding continuity equation is

$$\ddot{\phi}(f_T - \dot{\phi}^2 f_{TT}) + 3H\dot{\phi}f_T = 0 \quad (4.25)$$

or equivalently, in terms of T ,

$$(f_T + 2T f_{TT})\dot{T} + 6HT f_T = 0, \quad (4.26)$$

where $\rho' = 2T f_T - f$, $p' = f$ and $\dot{\rho}' + 3H(\rho' + p') = 0$. Now let us split the equation (2.13) into two equations as

$$4\dot{H}T f_{TT} - (T - 2\dot{H})f_T = 0 \quad (4.27)$$

and

$$-4\dot{H} + T - f = 2k^2 p_m. \quad (4.28)$$

Eq.(4.27) satisfies automatically since it is just the another form of the continuity equation (4.26). So finally the system of equations of $F(T)$ - gravity takes the form

$$-T - 2T f_T + f = 2k^2 \rho_m, \quad (4.29)$$

$$-4\dot{H} + T - f = 2k^2 p_m, \quad (4.30)$$

$$(f_T + 2T f_{TT})\dot{T} + 6HT f_T = 0, \quad (4.31)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (4.32)$$

It transforms to the equations (4.3)-(4.6) after the identifications $T = X = -6H^2$ and $f = 2k^2 K$. So we can conclude that for the special case (4.24) both $F(T)$ - gravity and purely kinetic k-essence are equivalent to each other at least in the equation's level. Some comments on the continuity equation (4.25) [= (4.26) = (4.27)]. It has two integrals of motion ($I_{jT} = 0$):

$$I_1 = a_0^{-3} a^3 T^{0.5} f_T, \quad I_2 = f - a^3 T^{0.5} f_T \partial_T^{-1} (a^{-3} T^{-0.5}). \quad (4.33)$$

Its general solution is given by

$$f = C_2 + iC_1 a_0^2 \partial_T^{-1} (a^{-3} T^{-0.5}), \quad C_j = \text{const}. \quad (4.34)$$

Finally we would like to present the exact solution both $F(T)$ -gravity and purely kinetic k-essence. As an example let us consider Λ CDM for which $a^{-3} = -\frac{1}{2\rho_0}(T + 2\Lambda) = -\frac{1}{2\rho_0}(X + 2\Lambda)$ so that

$$f = f(X) = f(T) = C_2 - \frac{iC_1 a_0^3}{3\rho_0} (T^{1.5} + 6\Lambda T^{0.5}) = C_2 - \frac{iC_1 a_0^3}{3\rho_0} (X^{1.5} + 6\Lambda X^{0.5}), \quad (4.35)$$

which is the M_{32} - model. It is the exact solution of the equations of motion of purely kinetic k-essence and $F(T)$ - gravity simultaneously.

5 Conclusion

In this work we investigated the recently developed $F(T)$ gravity, which is a new modified gravity capable of accounting for the present cosmic accelerating expansion with no need of dark energy. $F(T)$ gravity as the modified teleparallel gravity is the extension of the "teleparallel" equivalent of General Gravity (TEGR), which uses the zero curvature Weitzenböck connection instead of the torsionless Levi-Civita connection, in the same lines as $F(R)$ gravity is the extension of standard General Gravity. In particular, we presented some new models of $F(T)$ gravity. We analyze the relation between $F(T)$ gravity and k-essence. We also studied some new models of k-essence namely some parametric models of purely kinetic k-essence.

6 Appendix A: Multiple k-essence

For the multiple k-essence the action reads as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + K(X_1, X_2, \dots, X_n, \phi_1, \phi_2, \dots, \phi_n) + L_m \right]. \quad (6.1)$$

The corresponding closed set of equations reads as

$$3k^{-2}H^2 = 2 \sum_{j=1}^n X_j K_{X_j} - K + \rho_m, \quad (6.2)$$

$$-k^{-2}(2\dot{H} + 3H^2) = K + p_m, \quad (6.3)$$

$$(K_{X_j} + 2X_j K_{X_j X_j})\dot{X}_j + 6HX_j K_{X_j} - K_{\phi_j} = 0, \quad (6.4)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (6.5)$$

where $X_j = -0.5\dot{\phi}_j^2$. The k-essence energy density and pressure respectively, given by

$$\rho_j = 2X_j K_{X_j} - K, \quad p_j = w_j \rho_j. \quad (6.6)$$

The equations of motion of the scalar fields ϕ_j are given as

$$-(a^3 \dot{\phi}_j K_{X_j})_t = a^3 K_{\phi_j}. \quad (6.7)$$

Note that the case $K = \sum_{j=1}^n K_j(X_j, \phi_j)$ (mutually non-interacting scalar fields) was investigated in [38].

7 Appendix B: Some models of modified gravity theories and k-essence

In this Appendix B, we present some models of modified gravity theories and k-essence [$N = \ln a$, $R = 6(\dot{H} + 2H^2)$, $G = 24H^2(\dot{H} + H^2)$, $T = -6H^2$, $\eta = \int a^{-1} dt = \int H^{-1} a^{-2} da$, $\xi = \ln T$, $\zeta = \ln X$, $\varsigma = \ln R$, $\vartheta = \ln G$].

1) The M_1 - model. Its Lagrangian has the form

$$F = K = \sum_{j=-m}^n \nu_j(t) t^j. \quad (7.1)$$

2) The M_2 - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) e^{jt}. \quad (7.2)$$

3) The M_3 - model. It corresponds to the Lagrangian

$$F = K = \sum_{j=-m}^n \nu_j(t) \tanh[jt]. \quad (7.3)$$

4) The M_4 - model. The Lagrangian of this model is given by

$$F = K = \sum_{j=-m}^n \nu_j(t) \tanh[t]^j. \quad (7.4)$$

5) The M_5 - model. The Lagrangian is given by

$$F = K = \sum_{j=-m}^n \nu_j(t) \cosh[t]^j. \quad (7.5)$$

6) The M_6 - model. It reads as

$$K = \sum_{j=-m}^n \nu_j(t) \tan[t]^j. \quad (7.6)$$

7) The M_7 - model. It reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) \cos[t]^j. \quad (7.7)$$

8) The M_8 - model. It reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) \cosh[jt]. \quad (7.8)$$

9) The M_9 - model. The corresponding Lagrangian reads as

$$K = \sum_{j=-m}^n \nu_j(t) t^j e^{jt}. \quad (7.9)$$

10) The M_{10} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) H^j. \quad (7.10)$$

11) The M_{11} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) a^j. \quad (7.11)$$

12) The M_{12} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) N^j. \quad (7.12)$$

13) The M_{13} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) T^j. \quad (7.13)$$

14) The M_{14} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) G^j. \quad (7.14)$$

15) The M_{15} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) R^j. \quad (7.15)$$

16) The M_{16} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{i,j} \nu_{ij}(t) R^i G^j. \quad (7.16)$$

17) The M_{17} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{i,j} \nu_{ij}(t) R^i T^j. \quad (7.17)$$

18) The M_{18} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{i,j} \nu_{ij}(t) T^i G^j. \quad (7.18)$$

19) The M_{19} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) e^{jN}. \quad (7.19)$$

20) The M_{20} -model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) \eta^j. \quad (7.20)$$

21) The M_{21} -model [see e.g. (2.33)].

22) The M_{22} -model [see e.g. (2.41)].

23) The M_{23} -model [see e.g. (3.23)].

24) The M_{24} -model [see e.g. (3.24)].

25) The M_{25} -model [see e.g. (2.48)].

26) The M_{26} -model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) \zeta^j. \quad (7.21)$$

27) The M_{27} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) \vartheta^j. \quad (7.22)$$

28) The M_{28} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) (\ln \eta)^j. \quad (7.23)$$

29) The M_{29} - model. The corresponding Lagrangian reads as

$$F = K = \sum_{j=-m}^n \nu_j(t) (\tanh[\eta])^j. \quad (7.24)$$

30) The M_{30} - model. The corresponding Lagrangian reads as

$$K = \sum_{j=-m}^n \nu_j(t)(\ln[t])^j. \quad (7.25)$$

31) The M_{31} - model. The corresponding Lagrangian reads as

$$K = \sum_{j=-m}^n \nu_j(t)(\cosh[R])^j. \quad (7.26)$$

32) The M_{32} - model. The corresponding Lagrangian reads as

$$K = \sum_{j=-m}^n \nu_j(t)X^j. \quad (7.27)$$

8 Appendix C: Modified gravity theories as the particular reductions of purely kinetic k-essence

In this Appendix, we show that some important modified gravity theories, namely, $F(G)$, $F(R)$ and $F(T)$ can be written as the particular reductions of purely kinetic k-essence.

8.1 $F(G)$ gravity

8.1.1 Variant-I

Let us consider the following transformation (see, e.g. [3]- [13])

$$K = 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G + f - Gf_G, \quad (8.1)$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2. \quad (8.2)$$

Substituting these expressions e.g. into Eqs.(4.3)-(4.6) we get

$$0 = -3k^{-2}H^2 + Gf_G - f - 24\dot{G}H^3f_{GG} + \rho_m, \quad (8.3)$$

$$0 = 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G + k^{-2}(2\dot{H} + 3H^2) + f - Gf_G + p_m, \quad (8.4)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (8.5)$$

where

$$G = 24H^2(\dot{H} + H^2). \quad (8.6)$$

It is the system of the equations of motion of $F(G)$ -gravity with the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2k^2} R + f(G) + L_m \right]. \quad (8.7)$$

Now let us consider the particular case when $K = f$, $X = G$. Then instead of Eqs.(8.3)-(8.5) we obtain the following system

$$0 = -3k^{-2}H^2 + 2Gf_G - f + \rho_m, \quad (8.8)$$

$$0 = k^{-2}(2\dot{H} + 3H^2) + f + p_m, \quad (8.9)$$

$$(f_G + 2Gf_{GG})\dot{G} + 6HGf_G = 0, \quad (8.10)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \quad (8.11)$$

and

$$Gf_G + 24\dot{G}H^3f_{GG} = 0, \quad (8.12)$$

$$0 = 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G - Gf_G, \quad (8.13)$$

$$\kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 = 24H^2(\dot{H} + H^2). \quad (8.14)$$

Let's make two steps back that is let's simplify a problem: 1) we want reduce the problem to the case $\rho_m = p_m = 0$; 2) we want illustrate our results on the pedagogical example: $f(G) = \alpha G^n$. As result, the k-fields take the more simple form

$$K = \alpha(n-1)G^{n-3}\{8nH^2[(n-2)\dot{G} + G\ddot{G}] + 16nH(\dot{H} + H^2)G\dot{G} - G^3\}, \quad (8.15)$$

$$X = \kappa^{-1}k^{-4}a^6\dot{H}^2, \quad (8.16)$$

$$\phi = \pm i\sqrt{2\kappa^{-1}k^{-2}}\partial_t^{-1}(a^3\dot{H}). \quad (8.17)$$

Here we want to construct two examples of induced purely kinetic k-essence models: one is in the standard "canonical" form that means in the form $K = K(X)$ and another in the parametric form that means in the form $K = K(t)$, $X = X(t)$ (t plays the role of the parameter).

i) Let $a = \beta t^n$. Then the corresponding purely kinetic k-essence reads as

$$K = K(X) = 16\alpha n^9(n-1)^3[(113-33n)n^3(n-1)(\frac{X}{\gamma})^{\frac{8}{2-3n}} - 8(n-2)(\frac{X}{\gamma})^{\frac{11}{4-6n}}]. \quad (8.18)$$

Such model we call the "canonical" k-essence model. Note that for this case

$$X = \gamma t^{6n-4}, \quad \phi = \phi_0 + \frac{i\sqrt{2\gamma}}{3n-1}t^{3n-1}, \quad \gamma = \kappa^{-1}k^{-4}\beta^6n^2. \quad (8.19)$$

ii) Now we want to present the parametric k-essence model. To do it, let us consider an example: $H = \lambda t^m$. In this case the purely kinetic k-essence equivalent counterpart of the corresponding $F(G)$ -model is given by

$$K = K(t) = 24^{n-3}\alpha\lambda^{3(n-3)}(n-1)t^{(3m-1)(n-3)}[m + \lambda t^{m+1}]^{n-3}[K_1 + K_2], \quad (8.20)$$

$$X = X(t) = \kappa^{-1}k^{-4}a_0^6n^2\lambda^2t^{2(m-1)}e^{[\frac{6\lambda}{m+1}t^{m+1}]}, \quad (8.21)$$

where

$$K_1 = 192nm(n-2)\lambda^5t^{5m-2}[3m-1+4\lambda t^{m+1}] + 4608nm\lambda^8t^{8m-4}[m + \lambda t^{m+1}][(3m-1)(3m-2) + 4(4m-1)\lambda t^{m+1}], \quad (8.22)$$

$$K_2 = 9216nm\lambda^8t^{8m-4}[m+t][m + \lambda t^{m+1}][3m-1+4\lambda t^{m+1}] - 13824\lambda^9t^{9m-3}[m + \lambda t^{m+1}]^3. \quad (8.23)$$

Such model we call the parametric k-essence model.

8.1.2 Variant-II

We now introduce two functions K and X as

$$K = 8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G + f - Gf_G, \quad X = \frac{8H^2\ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G - 24\dot{G}H^3f_{GG}}{4\kappa a^{-6}}, \quad (8.24)$$

where $f(G)$ obeys the system (8.3)-(8.5). Then these functions solve the system of the equations of motion of purely kinetic k-essence (4.3)-(4.6).

8.2 $F(R)$ gravity

8.2.1 Variant-I

In this subsection we consider the following transformation (see, e.g. [3]- [13])

$$K = 2[\ddot{f}_R + 2H\dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R], \quad (8.25)$$

$$X = \kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2, \quad (8.26)$$

where

$$R = 6(\dot{H} + 2H^2). \quad (8.27)$$

The substitution (8.25)-(8.26) into Eqs.(4.3)-(4.6) gives

$$0 = -3k^{-2}H^2 - 6H\dot{R}f_{RR} + 6(\dot{H} + H^2)f_R - f + \rho_m, \quad (8.28)$$

$$0 = 2[\ddot{f}_R + 2H\dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R] + k^{-2}(2\dot{H} + 3H^2) + p_m, \quad (8.29)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (8.30)$$

It is the equations of $F(R)$ -gravity. The corresponding action is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2k^2} R + f(R) + L_m \right]. \quad (8.31)$$

Now let us consider the particular case when $K = f$, $X = R$. The corresponding continuity equation is $(f_R + 2Rf_{RR})\dot{R} + 6HRf_R = 0$. Then instead Eqs.(8.28)-(8.30) we obtain the system

$$0 = -3k^{-2}H^2 + 2Rf_R - f + \rho_m, \quad (8.32)$$

$$0 = k^{-2}(2\dot{H} + 3H^2) + f + p_m, \quad (8.33)$$

$$(f_R + 2Rf_{RR})\dot{R} + 6HRf_R = 0, \quad (8.34)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \quad (8.35)$$

and

$$0 = -6H\dot{R}f_{RR} + 6(\dot{H} + H^2)f_R - 2Rf_R, \quad (8.36)$$

$$0 = \ddot{f}_R + 2H\dot{f}_R - (\dot{H} + 3H^2)f_R, \quad (8.37)$$

$$\kappa^{-1}k^{-4}a^6[\dot{H} + 0.5k^2(\rho_m + p_m)]^2 = 6(\dot{H} + 2H^2). \quad (8.38)$$

Let us construct an example of the purely kinetic k-essence model induced by $F(R)$ - gravity. Let's simplify a problem: we assume that $\rho_m = p_m = 0$ and $f(R) = \alpha R^n$. As result, the k-essence Lagrangian takes the form

$$K = 2\alpha R^{n-3}[n(n-1)(n-2)\dot{R}^2 + n(n-1)R\ddot{R} + 2n(n-1)HR\dot{R} + 0.5R^3 - n(\dot{H} + 3H^2)R^2]. \quad (8.39)$$

Here

$$X = \kappa^{-1}k^{-4}a^6\dot{H}^2, \quad \phi = \pm i\sqrt{2\kappa^{-1}k^{-2}}\partial_t^{-1}(a^3\dot{H}). \quad (8.40)$$

Now we construct the model for the case $a = \beta t^l$. Then the corresponding purely kinetic k-essence reads as

$$K = K(X) = \varsigma X^{\frac{n}{2-3l}}, \quad (8.41)$$

where

$$\varsigma = 72\alpha l(2l-1)^{n-1}(6l)^{n-3}[2ln(n-1)(2n-2l-1) + 3(2l-1) - nl^2(3l-1)^2]\gamma^{\frac{n}{3l-2}}. \quad (8.42)$$

Note that X and ϕ are given by

$$X = \gamma t^{6l-4}, \quad \phi = \phi_0 + \frac{i\sqrt{2\gamma}}{3l-1}t^{3l-1}, \quad \gamma = \kappa^{-1}k^{-4}\beta^6l^2. \quad (8.43)$$

Similarly we can construct a new class k-essence models induced by modified gravity theories. These new k-essence models give the equivalent descriptions of dark energy/matter.

8.2.2 Variant-II

If we introduce the following two functions K and X

$$K = 2[\ddot{f}_R + 2H\dot{f}_R + 0.5f - (\dot{H} + 3H^2)f_R], \quad X = \frac{2[\ddot{f}_R + 2H\dot{f}_R] - 6H\dot{R}f_{RR} + 4\dot{H}f_R}{4\kappa a^{-6}}, \quad (8.44)$$

then they satisfy the equations (4.3)-(4.6).

8.3 $F(R, G)$ gravity

The action of $F(R, G)$ - gravity is given (see, e.g. [3]- [13])

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2k^2} F(R, G) + L_m \right]. \quad (8.45)$$

The corresponding system of equations is given by

$$3k^{-2}H^2 = \rho_{eff}, \quad (8.46)$$

$$-k^{-2}(2\dot{H} + 3H^2) = p_{eff}, \quad (8.47)$$

$$\dot{\rho}_{eff} + 3H(\rho_{eff} + p_{eff}) = 0, \quad (8.48)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (8.49)$$

Here

$$\rho_{eff} = \frac{1}{F_R} \{ \rho_m + 0.5k^{-2} [RF_R - F - 6H\dot{F}_R + GF_G - 24H^3\dot{F}_G] \}, \quad (8.50)$$

$$p_{eff} = \frac{1}{F_R} \{ p_m + 0.5k^{-2} [-RF_R + F + 4H\dot{F}_R + 2\ddot{F}_R - GF_G + 16H(\dot{H} + H^2)\dot{F}_G + 8H^2\ddot{F}_G] \}. \quad (8.51)$$

Let us consider the following transformation

$$K = \frac{1}{F_R} \{ p_m + 0.5k^{-2} [-RF_R + F + 4H\dot{F}_R + 2\ddot{F}_R - GF_G + 16H(\dot{H} + H^2)\dot{F}_G + 8H^2\ddot{F}_G] \}, \quad (8.52)$$

$$X = 0.25\kappa^{-1}a^6F_R^{-1} \{ \rho_m + p_m + k^{-2} [\ddot{F}_R - H\dot{F}_R + 4H(2\dot{H} - H^2)\dot{F}_G + 4H^2\ddot{F}_G] \}. \quad (8.53)$$

After this transformation, Eqs.(8.46)-(8.49) take the form (4.3)-(4.6). It is the system of equations of purely kinetic k-essence. So in this sense, both $F(R, G)$ - gravity and purely kinetic k-essence is equivalent to each other. Hence follow the results of the previous two subsections. In fact $F(G)$ and $F(R)$ are the particular reductions of $F(R, G)$ e.g. as: $F(R) = F(R, 0)$ and $F(G) = F(0, G)$.

9 Appendix D: Some generalized gas models

One of interesting class of gas/fluid models is models induced by elliptic functions (see also Refs.[39]-[42]). Here some of such models [Below, $\sigma(\rho)$ is the Weierstrass $\sigma(x)$ - function, $\zeta(x)$ is the Weierstrass $\zeta(x)$ - function, $am(x)$ is the Jacobi amplitude ($am(x)$) function and so on]. Table 1.

$MG - I$	$model$	$H = \zeta(t)$
$MG - II$	$model$	$a = \zeta(t)$
$MG - III$	$model$	$H = \sigma(t)$
$MG - IV$	$model$	$a = \sigma(t)$
$MG - V$	$model$	$H = \text{cn}'t$
$MG - VI$	$model$	$H = \text{sn}'t$
$MG - VII$	$model$	$H = \text{dn}'t$
$MG - VIII$	$model$	$H = \text{cnt}$
$MG - IX$	$model$	$H = \text{snt}$
$MG - X$	$model$	$H = \text{dnt}$

Table 2.

$MG - XI$	$model$	$p = -B[\zeta(\rho)]^\alpha$
$MG - XII$	$model$	$H = \wp(t)$
$MG - XIII$	$model$	$H = \wp(t)'$
$MG - XIV$	$model$	$H = \wp(t)''$
$MG - XV$	$model$	$H = \wp(t)'''$
$MG - XVI$	$model$	$H = \wp(t)^{IV}$
$MG - XVII$	$model$	$a(t) = \wp(t)$
$MG - XVIII$	$model$	$a = \wp(t)'$
$MG - XIX$	$model$	$a = \wp(t)''$
$MG - XX$	$model$	$a = \wp(t)'''$

Table 3.

<i>MG - XXI model</i>	$p = -B[\wp(\rho)]^{0.5}$
<i>MG - XXII model</i>	$p = -B[\wp(\rho)]^{0.5\alpha}$
<i>MG - XXIII model</i>	$p = A\rho - B[\wp(\rho)]^{0.5\alpha}$
<i>MG - XXIV model</i>	$p = A\sigma(\rho) - B[\sigma(\rho)]^{-\alpha}$
<i>MG - XXV model</i>	$p = \frac{A}{\zeta(\rho)} - B[\zeta(\rho)]^\alpha$
<i>MG - XXVI model</i>	$p = A[\wp(\rho)]^{-0.5} - B[\wp(\rho)]^{0.5\alpha}$
<i>MG - XXVII model</i>	$p = A[am(\rho)] - B[am(\rho)]^{-\alpha}$
<i>MG - XXVIII model</i>	$p = A\sigma(\rho)$
<i>MG - XXIX model</i>	$p = -B[\sigma(\rho)]^{-\alpha}$
<i>MG - XXX model</i>	$p = \frac{A}{\zeta(\rho)}$
<i>MG - XXXI model</i>	$p = A[am(\rho)]$
<i>MG - XXXII model</i>	$p = -B[am(\rho)]^{-\alpha}$

(9.3)

10 Appendix E: Knot Universes from Bianchi type-I models

Our aim in this Appendix is to present some simplest examples of knot universes for the Bianchi type - I model. The corresponding metric reads as

$$ds^2 = -dt^2 + A^2 dx_1^2 + B^2 dx_2^2 + C^2 dx_3^2, \quad (10.1)$$

where we assume that t, x_i, A, B, C are dimensionless and A, B, C are functions of t alone. For the metric (10.1) it is well-known that the field equations take the form

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \rho = 0, \quad (10.2)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + p_1 = 0, \quad (10.3)$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} + p_2 = 0, \quad (10.4)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + p_3 = 0, \quad (10.5)$$

where we assume that $p_1 \neq p_2 \neq p_3$. Consider some examples.

10.1 The trefoil knot universe

Let us consider the following solution of the system (10.2)-(10.5)

$$A = [2 + \cos(3t)] \cos(2t), \quad (10.6)$$

$$B = [2 + \cos(3t)] \sin(2t), \quad (10.7)$$

$$C = \sin(3t) \quad (10.8)$$

and the corresponding expressions for ρ and p_i . The solution (10.6)-(10.8) is the parametric equation of the trefoil knot. For that reason the corresponding universe we call the trefoil knot universe.

10.2 The figure-eight knot universe

Our second example is given by the following solution of the system (10.2)-(10.5)

$$A = [2 + \cos(2t)] \cos(3t), \quad (10.9)$$

$$B = [2 + \cos(2t)] \sin(3t), \quad (10.10)$$

$$C = \sin(4t). \quad (10.11)$$

and the corresponding expressions for ρ and p_i . For this solution the corresponding universe we call the figure-eight knot universe as the solution (10.9)-(10.11) is nothing but the parametric equation of the figure-eight knot (see also Refs. [43]-[44]).

11 Appendix J: Some generalizations of $F(T)$ gravity

In this section we present 3 generalizations of the Friedmann equations of $F(T)$ gravity. Recall that the modified Friedmann equations of the loop quantum cosmology (LQC) read as

$$3H^2 = 8\pi G\rho \left(1 - \frac{\rho}{\rho_c}\right), \quad (11.1)$$

$$\dot{H} = -4\pi G(\rho + p) \left(1 - \frac{2\rho}{\rho_c}\right), \quad (11.2)$$

$$\dot{\rho} = -3H(\rho + p). \quad (11.3)$$

11.1 The M_{35} - model

The M_{35} - model is the given by

$$3H^2 = 8\pi G\rho \sqrt{1 - \frac{2\rho}{\rho_c}}, \quad (11.4)$$

$$\dot{H} = -\frac{4\pi G(\rho + p)}{\sqrt{1 - \frac{2\rho}{\rho_c}}} \left(1 - \frac{3\rho}{\rho_c}\right), \quad (11.5)$$

$$\dot{\rho} = -3H(\rho + p). \quad (11.6)$$

11.2 The M_{36} - model

For the FRW metric the equations of the M_{36} - model read as

$$3H^2 = 8\pi G\rho_c \sqrt{1 - \frac{2\rho}{\rho_c}} \left(1 - \sqrt{1 - \frac{2\rho}{\rho_c}}\right), \quad (11.7)$$

$$\dot{H} = -8\pi G(\rho + p) \left(1 - \frac{1}{2\sqrt{1 - \frac{2\rho}{\rho_c}}}\right), \quad (11.8)$$

$$\dot{\rho} = -3H(\rho + p). \quad (11.9)$$

In the case $2\rho < \rho_c$ we have $\sqrt{1 - \frac{2\rho}{\rho_c}} \approx 1 - \frac{\rho}{\rho_c}$ so that the previous two systems (11.4)-(11.6) and (11.7)-(11.9) transforms to the usual system of equations of LQC (11.1)-(11.3).

11.3 The M_{37} - model

Let us consider the M_{37} - model. Its action is

$$\mathcal{S} = \int d^4x e [F(R, T) + L_m], \quad (11.10)$$

where R is the "curvature" scalar and T is the "torsion" scalar. For simplicity in this paper we work in such FRW spacetime in which R and T are given by

$$R = u + 6(\dot{H} + 2H^2), \quad (11.11)$$

$$T = v - 6H^2, \quad (11.12)$$

where in general $u = u(t, a, \dot{a}, \ddot{a}, \dots; f_i)$ and $v = v(t, a, \dot{a}, \ddot{a}, \dots; g_i)$ are some real functions, $H = (\ln a)_t$, f_i and g_i are some unknown functions related with the geometry of the spacetime. Here we restrict ourselves to the case when $u = u(a, \dot{a})$ and $v = v(a, \dot{a})$. Then the FRW equations of the M_{37} - model look like

$$D_2 F_{RR} + D_1 F_R + J F_{RT} + E_1 F_T + K F = -a^3 \rho, \quad (11.13)$$

$$U + C_2 F_{RRT} + C_1 F_{RTT} + C_0 F_{RT} + M F = -3a^2 p, \quad (11.14)$$

$$\dot{\rho} - 3H(\rho + p) = 0, \quad (11.15)$$

where

$$U = A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R + B_2 F_{TT} + B_1 F_T. \quad (11.16)$$

Here

$$D_2 = -6\dot{R}a^2\dot{a}, \quad (11.17)$$

$$D_1 = -6a\dot{a}^2 + a^3 u_{\dot{a}}\dot{a} - a^3(u - R), \quad (11.18)$$

$$J = -6a^2\dot{a}\dot{T}, \quad (11.19)$$

$$E_1 = -6a\dot{a}^2 + a^3 v_{\dot{a}}\dot{a} - a^3(v - T), \quad (11.20)$$

$$K = -a^3 \quad (11.21)$$

and

$$A_3 = -6\dot{R}^2a^2, \quad (11.22)$$

$$A_2 = -6\ddot{R}a^2 - 12\dot{R}a\dot{a} + a^3\dot{R}u_{\dot{a}}, \quad (11.23)$$

$$A_1 = -6\dot{a}^2 - 12a\ddot{a} + 3a^2\dot{a}u_{\dot{a}} + a^3\dot{u}_{\dot{a}} - 3a^2(u - R) - a^3u_a, \quad (11.24)$$

$$B_2 = -12\dot{T}a\dot{a} + a^3\dot{T}v_{\dot{a}}, \quad (11.25)$$

$$B_1 = -6\dot{a}^2 - 12a\ddot{a} + 3a^2\dot{a}v_{\dot{a}} + a^3\dot{v}_{\dot{a}} - 3a^2(v - T) - a^3v_a, \quad (11.26)$$

$$C_2 = -12a^2\dot{R}\dot{T}, \quad (11.27)$$

$$C_1 = -6a^2\dot{T}^2, \quad (11.28)$$

$$C_0 = -12\dot{R}a\dot{a} - 12\dot{T}a\dot{a} - 6a^2\ddot{T} + a^3\dot{R}v_{\dot{a}} + a^3\dot{T}u_{\dot{a}}, \quad (11.29)$$

$$M = -3a^2. \quad (11.30)$$

The system (11.13)-(11.15) admit two important reductions. Let us now present these particular cases.

a) *Case:* $F = F(T)$, $u = v = 0$. Then the system (11.13)-(11.15) becomes

$$E_1 F_T + K F = -a^3 \rho, \quad (11.31)$$

$$B_2 F_{TT} + B_1 F_T + M F = -3a^2 p, \quad (11.32)$$

$$\dot{\rho} - 3H(\rho + p) = 0. \quad (11.33)$$

This system we can rewrite as

$$-2TF_T + F = \rho, \quad (11.34)$$

$$-8\dot{H}TF_{TT} + 2(T - 2\dot{H})F_T - F = p, \quad (11.35)$$

$$\dot{\rho} - 3H(\rho + p) = 0 \quad (11.36)$$

that is nothing but the system of $F(T)$ gravity.

b) *Case:* $F = F(R)$, $u = v = 0$. Second reduction we get if consider the case when $F = F(R)$, $u = v = 0$. Then the system (11.13)-(11.15) reads as

$$D_2 F_{RR} + D_1 F_R + K F = -a^3 \rho, \quad (11.37)$$

$$A_3 F_{RRR} + A_2 F_{RR} + A_1 F_R + M F = -3a^2 p, \quad (11.38)$$

$$\dot{\rho} - 3H(\rho + p) = 0, \quad (11.39)$$

where

$$A_3 = -6\dot{R}^2a^2, \quad (11.40)$$

$$A_2 = -6\ddot{R}a^2 - 12\dot{R}a\dot{a}, \quad (11.41)$$

$$A_1 = -6\dot{a}^2 - 12a\ddot{a} + 3a^2R, \quad (11.42)$$

$$D_2 = -6\dot{R}a^2\dot{a}, \quad (11.43)$$

$$D_1 = -6a\dot{a}^2 + a^3R, \quad (11.44)$$

$$K = -a^3. \quad (11.45)$$

This system can be written as

$$6\dot{R}H F_{RR} - (R - 6H^2)F_R + F = \rho, \quad (11.46)$$

$$-2\dot{R}^2 F_{RRR} + [-4\dot{R}H - 2\ddot{R}]F_{RR} + [-2H^2 - 4a^{-1}\ddot{a} + R]F_R - F = p, \quad (11.47)$$

$$\dot{\rho} - 3H(\rho + p) = 0. \quad (11.48)$$

It is nothing but the system of equations of $F(R)$ gravity. So we have shown that our model contains $F(R)$ and $F(T)$ gravity models as particular cases. In this sense the M_{37} - model is the generalization of these two known modified gravity theories.

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